TAF 2-25-11 Bailey

Note Title 2/25/2011

Recall:

$$\mathrm{Hom}(T, A^\vee) = \left\{ \begin{array}{l} \text{iso classes of} \\ \text{invertible sheaves} \\ \text{over } A \times T \text{ having trivial} \\ \text{restrictions } \{0\} \times T,\ A \times \{0\} \end{array} \right\}$$

Ex $\mathcal{L}$ invert. sheaf on A

$m_A^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1} \otimes p_2^* \mathcal{L}^{-1}$ is invertible over $A \times A$ restricts to trivial on $\{0\} \times A$ and $A \times \{0\}$

$\lambda_{\mathcal{L}} \in \mathrm{Hom}(A, A^\vee)$ symmetric

Converse If $\alpha \in \mathrm{Hom}(A, A^\vee)$ symmetric
$\implies \exists\, \mathcal{L}$ over A such that $\alpha = \lambda_{\mathcal{L}}$
$\mathcal{L}^2 = \Delta^* \mathcal{J}$

A polarization is a symm isogeny $\lambda : A \to A^\vee$
$(\lambda^\vee = \lambda)$

Weil pairing $\quad e_m : A_m(\bar{k}) \times A^\vee_m(\bar{k})$

$$\downarrow$$

$$\mu_m(\bar{k})$$

m-torsion $A(\bar{k})[m]$

nondegen commuting w/ action of $\mathrm{Gal}(\bar{k}/k)$

Suppose $\lambda : A \to A^\vee$ is a polarization.

$e_m^\lambda : A(\bar{k})[m] \times A(\bar{k})[m] \longrightarrow \mu_m \ \bar{k}$

$$e_m^\lambda(a,b) = e_m(a, \lambda b)$$

Let $m = \ell^n$ and take inverse limit

Weil-pairing $\lambda\langle\ ,\ \rangle : T_\ell(A) \times T_\ell(A) \longrightarrow \mathbb{Z}_\ell(1) = \varprojlim \mu_{\ell^n}(\bar{k})$

$\qquad\qquad '' \qquad : V_\ell(A) \times V_\ell(A) \longrightarrow \mathbb{Q}_\ell(1)$

nondegen since λ is an isogeny

(can be shown to be alternating)

Rosati involution RI

Fix a polarization $\lambda : A \to A^\vee$

$\lambda \in \mathrm{Hom}(A, A^\vee) \otimes \mathbb{Q}$ $\qquad \lambda^{-1} \in \mathrm{Hom}(A^\vee, A) \otimes \mathbb{Q}$

$\alpha \longmapsto \lambda^{-1} \circ \alpha^\vee \circ \lambda = \alpha^\dagger$ dagger

The Rosati invol. on $\mathrm{End}^0(A)$ corresponding to λ.

Note ① $(\alpha + \beta)^\dagger = \alpha^\dagger + \beta^\dagger$

$(\alpha\beta)^\dagger = \beta^\dagger \alpha^\dagger$

$\alpha^\dagger = \alpha$ for $\alpha \in \mathbb{Q}$

② For any $a, a' \in T_\ell(A) \otimes \mathbb{Q}$ $\qquad \ell \neq \mathrm{char}\, k$

$$e_\ell^\lambda(\alpha a, a') = e_\ell(\alpha a, \lambda a') = e_\ell(a, \alpha^\vee \circ \lambda a') = e_\ell^\lambda(a, \lambda^{-1} \circ \alpha^\vee \circ \lambda a')$$

$= e_\ell^{\lambda}(a, \alpha^\dagger a')$ and $\alpha^{\dagger\dagger} = \alpha$

③ If we replace λ with $n\lambda$ ($= \lambda \circ n$), the Rosati involution is unchanged!

$\Rightarrow$ RI depends only on the weak polarization class of λ.

Theorem The bilinear form $\mathrm{End}^0(A) \times \mathrm{End}^0(A) \longrightarrow \mathbb{Q}$

$$(\alpha, \beta) \longmapsto \mathrm{Tr}_{k/\mathbb{Q}}(\alpha \circ \beta^\dagger)$$

is positive definite.

$\mathrm{Tr}(\alpha \circ \alpha^\dagger) > 0$ if $\alpha \neq 0$

More precisely if $\mathcal{L}$ (or D) is an ample divisor defining λ of the RI then

$$\mathrm{Tr}\,(\alpha \circ \alpha^{\vee}) = \frac{2g}{(D^g)}\left(D^{g-1} \cdot \alpha^{*}(D)\right)$$

intersection #

Polarizations of B-linear abelian varieties.

A = abelian vty

B = simple $\mathbb{Q}$-algebra

Recall a B-linear abelian vty (A, i) is an ab vty A with embedding $B \xrightarrow{i} \mathrm{End}^0(A)$

Suppose $*$ is an invol on B

Def A polzn λ on (A, i) is <u>compatible</u> if the λ-RI restricts to $*$ on on B.

Note $*$ must be positive in order for a compat. polzn to exist.

Lemma If $*$ is a pos invol on B and A is B-linear then $\exists$ compat. polzn $\lambda : A \to A^\vee$.

$V_\ell(A) = T_\ell(A) \otimes \mathbb{Q}$ has a B-module structure.

$\lambda\langle \cdot , \cdot \rangle : V_\ell(A) \otimes_{\mathbb{Q}_\ell} V_\ell(A) \longrightarrow \mathbb{Q}_\ell(1)$

$$\lambda\langle bx, y\rangle = \lambda\langle x, b^* y\rangle \quad \text{so}$$

this pairing is $*$-Hermitian.

Induced polarizations

$\lambda_{\mathcal{L}} : A \longrightarrow A^\vee$ If $A' \xrightarrow{\alpha} A$ is an isogeny then

$$\begin{array}{ccc} A' & \xrightarrow{\alpha} & A \\ & & \downarrow \lambda \\ (A')^\vee & \xleftarrow{\alpha} & A^\vee \end{array}$$

this composite is $\alpha^* \lambda$, a polzn of A'

$$\alpha^* \mathcal{L} \longrightarrow \mathcal{L}$$
$$A' \xrightarrow{\alpha} A$$

Def If (A, λ) and (A', λ') are polarized ab vties, an isogeny $\alpha : A' \to A$ is an isometry if $\alpha^* \lambda = \lambda'$

Def If $\alpha^* \lambda$ is weakly equiv to λ' the α is a similitude.

<u>Lemma</u> Given (A,λ) and $\dagger$ the associated R I on $\mathrm{End}^0(A)$. An isogeny $\alpha: A \to A$

α is an isometry $\iff \alpha\alpha^\dagger = 1$

α is a similitude $\iff \alpha\alpha^\dagger \in \mathbb{Q}^\times$

Note $\lambda\langle\ ,\ \rangle : V_\ell(A) \otimes_{\mathbb{Q}_\ell} V_\ell(A) \to \mathbb{Q}_\ell(1)$

$$\lambda\langle \alpha_* x, \alpha_* y\rangle = \nu(\alpha)\,\lambda\langle x, y\rangle$$

Classification of weak polzns

char $(k) > p$

B = simple $\mathbb{Q}$-algebra

$*$ = positive invol on B

(A, i) = B-linear ab vty

λ = compat polzn

Alg gp over $\mathbb{Q}$

$$H_{(A,i,\lambda)}(R) = \{ h \in (\mathrm{End}^0_B(A) \otimes_{\mathbb{Q}} R)^\times : h^\dagger h \in R^\times)$$

__Ex__ $R = \mathbb{Q}$

$$H_{(A,i,\lambda)}(\mathbb{Q}) = \{ h \in \mathrm{End}^0_B(A) : h^\dagger h \in \mathbb{Q}^\times \}$$

$= \text{gp of similitudes of } \lambda.$

__Lemma__ $\left\{ \begin{array}{l} \text{similitude} \\ \text{classes of compat} \\ \text{weak polzns on } A \end{array} \right\} \longleftrightarrow \ker \left(\begin{array}{c} H^1(\mathbb{Q}, H_{(A,i,\lambda)}) \\ \downarrow \\ H^1(\mathbb{R}, \text{same}) \end{array} \right)$

If $\ell \neq p$

$$GU_{V_\ell(A)}(R) := \left\{ g \in \left(\mathrm{End}(V_\ell(A)) \otimes_{\mathbb{Q}_\ell} R \right)^\times \begin{array}{l} \exists \nu(g) \in R^\times \text{ such that} \\ \lambda\langle g(x), g(y) \rangle = \nu(g) \lambda \langle x, y \rangle \end{array} \right\}$$

similitude classes of nondegen $*$-Hermitian alternating form $V_\ell(A)$ classified by

$$H^1(\mathbb{Q}_\ell, GU_{V_\ell}(A))$$

$$H_{(A,i,\lambda)} \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{Q}_\ell)$$

$$\downarrow$$

$$GU_{V_\ell(A)}$$

$$H^1(\mathbb{Q}, H_{(A,i,\lambda)}) \longrightarrow H^1(\mathbb{Q}_\ell, GU_{V_\ell(A)})$$

Lemma For a compact polzn λ' on (A, i), the image of $[\lambda'] \in H^1(\mathbb{Q}; H_{(A,i,\lambda)})$ in $H^1(\mathbb{Q}, GU_{V_\ell(A)})$ quantifies the difference of similitude classes of alt. forms repr'd by $\lambda'\langle\ \rangle$ and $\lambda\langle\ \rangle$.